Solutions to Revision Exercise 2

1. (a) $\dfrac{1}{1+(x+1)^2}$ (b) $\dfrac{2x}{\sqrt{1-x^4}}$ (c) $\arcsin 2x + \dfrac{2x}{\sqrt{1-4x^2}}$

(d) $\dfrac{2\arcsin x}{\sqrt{1-x^2}}$ (e) $\dfrac{x-(1+x^2)\arctan x}{x^2(1+x^2)}$ (f) $\dfrac{1}{(1+4x^2)\sqrt{\arctan 2x}}$

(g) $\dfrac{1}{x[1+(\ln x)^2]}$ (h) $\dfrac{-x}{|x|\sqrt{1-x^2}}$.

2. (a) $y' = \dfrac{1}{4-3y^2}$ (b) $y' = \dfrac{y^2}{1+y^2}$ (c) $y' = \dfrac{1}{30(3y+2)^9}$

(d) $y' = \dfrac{-1}{3y^2-8y+3}$ (e) $y' = \dfrac{y^3}{-2\sin y + y\cos y}$ (f) $y' = \dfrac{2x(y+2)}{1-x^2}$

(g) $y' = -\dfrac{x}{y}$ (h) $y' = \dfrac{3-2x}{2y}$ (i) $y' = \dfrac{12x-y}{x+8y}$

(j) $y' = \dfrac{6xy-8x^2}{3y^2-3x^2}$ (k) $y' = \dfrac{2x\sin y + y\sin x}{\cos x - x^2\cos y}$ (l) $y' = \dfrac{\cos y + y^2\cos x}{x\sin y - 2y\sin x}$.

3. (a) $f'(x) = -2nx(1-x^2)^{n-1}$

$$\Rightarrow \quad (1-x^2)f'(x) + 2nxf(x) = -2nx(1-x^2)^n + 2nx(1-x^2)^n = 0 .$$

(b) Differentiate $(1-x^2)f'(x) + 2nxf(x) = 0$

$$\Rightarrow \quad (1-x^2)f''(x) - (2-2n)xf'(x) + 2nf(x) = 0$$

Differentiate again

$$\Rightarrow \quad (1-x^2)f'''(x) - (4-2n)xf''(x) + (4n-2)f'(x) = 0$$

Inductively, we get for any $k = 0, 1, 2, \ldots$

$$(1-x^2)f^{(k+1)}(x) - (2k-2n)xf^{(k)}(x) + k(3-k)f^{(k-1)}(x) = 0$$

Take $k = n+1$,

$$(1-x^2)f^{(n+2)}(x) - 2xf^{(n+1)}(x) - (n-2)(n+1)f^{(n)}(x) = 0 .$$

4. (a) $f(x) = e^x \ln(1+x)$

$$f'(x) = e^x \left[\ln(1+x) + \frac{1}{1+x} \right]$$

$$f''(x) = e^x \left[\ln(1+x) + \frac{2}{1+x} - \frac{1}{(1+x)^2} \right]$$

Therefore, $(1+x) f''(x) - (1+2x) f'(x) + x f(x)$

$$= e^x \Big[(1+x)\ln(1+x) + 2 - \frac{1}{1+x}$$

$$- (1+2x)\ln(1+x) - \frac{1+2x}{1+x} + x \ln(1+x) \Big]$$

$$= 0 .$$

(b) Differentiate $(1+x) f'' - (1+2x) f' + x f = 0$

$$\Rightarrow \quad (1+x) f''' - 2x f'' + (x-2) f' + f = 0$$

which is the case $n = 0$.

Assume the equality is true for $n = k$,

$$(1+x) f^{(k+3)} + (k-2x) f^{(k+2)} + (x-2k-2) f^{(k+1)} + (k+1) f^{(k)} = 0$$

differentiate in x,

$$\Rightarrow \quad (1+x) f^{(k+4)} + ((k+1)-2x) f^{(k+3)} + (x - 2(k+1) - 2) f^{(k+2)} + (k+2) f^{(k+1)} = 0$$

which is the case for $n = k+1$. So, the equality is true for all n by M.I.

5. (a) $f(x) = \dfrac{\ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}}$; $f'(x) = \dfrac{1 - x \dfrac{\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}}{1+x^2}$

$$\Rightarrow \quad (1+x^2) f'(x) + x f(x) = 1 .$$

(b) Differentiate (a), $\Rightarrow (1+x^2) f'' + 3x f' + f = 0$ ($n=0$ case).

Assume $n=k$ holds, i.e. $(1+x^2) f^{(k+2)} + (2k+3) x f^{(k+1)} + (k+1)^2 f^{(k)} = 0$

Differentiate $\Rightarrow (1+x^2) f^{(k+3)} + (2(k+1)+3) x f^{(k+2)} + (k+2)^2 f^{(k+1)} = 0 .$

so $n = k+1$ holds. So it holds for all n by M.I.

6. (a) $f(x) = \dfrac{2\arcsin x}{\sqrt{1-x^2}}$; $f'(x) = \dfrac{2 + 2\arcsin x \cdot \dfrac{x}{\sqrt{1-x^2}}}{1-x^2}$

$\Rightarrow \quad (1-x^2) f' - xf = 2$.

(b) Differentiate (a), $\Rightarrow \quad (1-x^2) f'' - 3x f' - f = 0 \qquad (n=0)$

Assume $n=k$ holds, i.e. $(1-x^2) f^{(k+2)} - (2k+3)\, x f^{(k+1)} - (k+1)^2 f^{(k)} = 0$

Differentiate $\Rightarrow \quad (1-x^2) f^{(k+3)} - (2(k+1)+3)\, x f^{(k+2)} - (k+2)^2 f^{(k+1)} = 0$.

So $n = k+1$ holds. Result follows by M.I .

7. By Mean Value Theorem, $f(6) - f(3) = f'(\xi)(6-3)$ for some $\xi \in (3,6)$

But $|f'(\xi) - 9| \le 3 \Rightarrow 6 \le f'(\xi) \le 12$

Hence, $\quad 18 \le f(6) - f(3) \le 36$.

8. (a) (i) $f'(x) = \beta x^{\beta-1} - \beta$

(ii) When $x \in (0,1)$, $f'(x) = \beta x^{\beta-1} - \beta < \beta - \beta = 0$

$\Rightarrow f$ is strictly decreasing on $(0,1]$.

(iii) When $x \in (1,+\infty)$, $f'(x) = \beta x^{\beta-1} - \beta > \beta - \beta = 0$.

$\Rightarrow f$ is strictly increasing on $[1,+\infty)$.

(iv) $\lim\limits_{x\to 0} f(x) = \beta - 1 > 0$; $\lim\limits_{x\to+\infty} f(x) = +\infty$.

So, $\min = 0$ at $x=1$ and no maximum.

(b) From (a), $x^\beta + \beta - 1 - \beta x \ge 0$ for all $x \in (0,+\infty)$.

Let $r = x-1$, then we have

$(1+r)^\beta + \beta - 1 - \beta(1+r) \ge 0$ for all $r \in (-1,+\infty)$

ie $\quad (1+r)^\beta \ge 1 + \beta r$ for all $r \in (-1,+\infty)$.

9. (a) Mean Value Thm $\Rightarrow$ $\arctan(x) - \arctan(0) = \dfrac{x}{1+\xi^2}$ for some $\xi \in (0,x)$

Note that $\dfrac{1}{1+x^2} < \dfrac{1}{1+\xi^2} < 1$ and $\arctan(0) = 0$.

$$\Rightarrow \quad \frac{x}{1+x^2} < \arctan x < x .$$

(b) By Taylor's Theorem, $\forall x \in (0,+\infty)$

$$\ln(1+x) - \frac{2x}{2+x} = \left[\frac{2}{(1+\xi)^3} - \frac{24}{(2+\xi)^4}\right]\frac{x^3}{6} \quad \text{for some } \xi \in (0,x).$$

One can check that

$$0 < \frac{2}{(1+x)^3} - \frac{24}{(2+x)^4} < \frac{1}{2} \qquad \forall x \in (0,\infty).$$

This yields the inequality

$$0 < \ln(1+x) - \frac{2x}{2+x} < \frac{x^3}{12} .$$

10. (a) By Taylor's Theorem, $\forall x \in (0,2\pi]$

$$\cos x = 1 + (-\cos\xi)\frac{x^2}{2} \quad \text{for some } \xi \in (0,x)$$

Since $-\cos\xi > -1$ for $\xi \in (0,2\pi)$, we have

$$\cos x > 1 - \frac{x^2}{2} . \qquad \forall x \in (0,2\pi]$$

(b) By Taylor, $\forall x \in (0,2\pi]$

$$\cos x = 1 - \frac{x^2}{2} + (\cos\xi)\frac{x^4}{24} \quad \text{for some } \xi \in (0,x).$$

Since $\cos\xi < 1$ for $\xi \in (0,2\pi)$, we have

$$\cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24} \qquad \forall x \in (0,2\pi].$$

(c) Since $1 - \dfrac{x^2}{2} < -1$ and $1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} > 1$ $\forall x \in (2\pi,+\infty)$

clearly, $1 - \dfrac{x^2}{2} < \cos x < 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$ $\forall x \in (2\pi,+\infty)$

(d) By (a) – (c), and that all 3 functions are even, we get

$$1 - \frac{x^2}{2} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24} \qquad \forall x \in \mathbb{R} \setminus \{0\}.$$

11. (a) 1 (b) 2 (c) 1 (d) 1 (e) $\frac{4}{5}$ (f) $\frac{1}{12}$

(g) 2 (h) $\frac{1}{2}$ (i) -30 (j) 1 (k) 0 (l) 2

(m) Not exist. (n) Not exist. (o) 1.

12. (a) 0 (b) 0 (c) $\frac{1}{2}$ (d) $-\frac{e}{2}$

13. (a) 0 (b) 0 (c) 1 (d) 0

14. (a) 1 (b) 1 (c) 1 (d) e (e) e^{-2}

(f) 1 (g) e^2 (h) e^2 (i) e^2 (j) 1

(k) e (l) 1 (m) e^{-e} (n) e^2 (o) 1

(p) 1 (q) $e^{1/12}$

15. (a) 1 (b) 1 (c) 0